

Question 1] Define Subgroup with examples.

prove, if G be a group and H a non-empty subset of G , then H is a subgroup of G if, and only if (i) for all $a \in H, b \in H \Rightarrow ab \in H$, i.e., H is closed under the given operation and (ii), for all $a \in H$, the inverse of a i.e. $a^{-1} \in H$.

Answer Subgroup (defn.): — Let G be a group and H a subset of G . Then H is called to be a subgroup of G if H is a group under the group operation of G .

Example (1) The set of integers I is an additive group. The set E of even integers is a subset of I and it is a group under addition.

Hence E is a subgroup of I .

Ex. (2) The set Q of rational numbers is a group under addition. The set I of integers is a subset of Q and is a group under addition.

Hence I is a subgroup of Q .

Proof of the theorem: — The conditions are necessary: Here the fact follows from the first part of the theorem that H is a group which is given under the group operation of G . Now since H is a group, therefore, $\forall a, b \in H$, we have $ab \in H$ and thus the first condition is satisfied.

Also, $\forall a \in H$, we have $a^{-1} \in H$ and thus the second axiom is satisfied.

Hence the first part of the theorem is proved.

The conditions are sufficient: Now, in order to prove the second part, we must prove that H is a group under the operation of G . We shall show with the given two conditions, that H satisfies all the four postulates of a group.

(i) We have, $\forall a, b \in H, ab \in H$

Therefore, the first postulate is satisfied.

(ii) Associativity: H is associative because H is a subset of G which is associative.

(iii) Suppose $x \in H$, then because of condition (ii), $x^{-1} \in H$ and hence the fourth postulate is satisfied.

(iv) Again, we have by (i) $xx^{-1} \in H$,

$$\Rightarrow e \in H,$$

Hence the identity e must exists in H .

Thus, we conclude that H satisfied all the four postulates of a groups and hence it is a group.

Hence the conditions are sufficient.

Hence the theorem proved.